Derivation of a general formula for $\sum_{i=1}^{n} i^k$, where $k \in \mathbb{N}$

There are many ways to derive a general formula for the geometric series. This paper will look at one such way. To find the formula for power $\boldsymbol{k}$ we can use $\sum_{i=1}^{n}(i+1)^{k+1}$, or alternatively $\sum_{i=1}^{n}(i-1)^{k+1}$.

We start by rewriting $\sum_{i=1}^{n}(i+1)^{k+1}$ as follows:

$$\sum_{i=1}^{n}(i+1)^{k+1} = \sum_{i=2}^{n+1} i^{k+1} = \sum_{i=1}^{n} i^{k+1} + (n+1)^{k+1} - 1 \tag{1}$$

To make the derivation look better/shorter (and more confusing) lets define $S_j[n]$ to be $\sum_{i=1}^{n} i^j$.

Using Binomial Theorem (see last page for definition) on LHS of (1):

$$\sum_{i=1}^{n}(i+1)^{k+1} = \sum_{i=1}^{n}\left(\sum_{j=0}^{k+1}\left[\frac{(k+1)!}{(k+1-j)!\,j!} i^j\right]\right) \tag{2}$$

Using the fact that we can interchange the summations:

$$\sum_{i=1}^{n}(i+1)^{k+1} = \sum_{j=0}^{k+1}\left(\sum_{i=1}^{n}\left[\frac{(k+1)!}{(k+1-j)!\,j!} i^j\right]\right) \tag{3}$$

Since $\frac{(k+1)!}{(k+1-j)!\,j!}$ is constant with respect to $\boldsymbol{i}$, then we can pull it out from the inner sum:

$$\sum_{i=1}^{n}(i+1)^{k+1} = \sum_{j=0}^{k+1}\left(\frac{(k+1)!}{(k+1-j)!\,j!}\left[\sum_{i=1}^{n} i^j\right]\right) \tag{4}$$

But $\left[\sum_{i=1}^{n} i^j\right]$, $S_j[n]$, is the sum formula for power $\boldsymbol{j}$; (4) becomes:

$$\sum_{i=1}^{n}(i+1)^{k+1} = \sum_{j=0}^{k+1}\left(\frac{(k+1)!}{(k+1-j)!\,j!} S_j[n]\right) \tag{5}$$

But we are looking for $S_k[n]$ and since the formula (5) depends on $S_k[n]$ and $S_{k+1}[n]$ (look at the summation limit) we would like to pull them out of the sum; (5) becomes:

$$\sum_{i=1}^{n}(i+1)^{k+1} = \sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!} S_j[n]\right) + \frac{(k+1)!}{(\cancel{k}+1-\cancel{k})!\,k!} S_k[n] + \frac{(k+1)!}{(\cancel{k}+\cancel{1}-\cancel{k}-\cancel{1})!(k+1)!} S_{k+1}[n]$$

$$= \sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!} S_j[n]\right) + \frac{(k+1)!}{k!} S_k[n] + \frac{(k+1)!}{(k+1)!} S_{k+1}[n]$$

$$= \sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!} S_j[n]\right) + (k+1)S_k[n] + \cancel{S_{k+1}[n]} = \cancel{S_{k+1}[n]} + (n+1)^{k+1} - 1$$

$$\Rightarrow \sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!}S_j[n]\right)+(k+1)S_k[n]=\left(n+1\right)^{k+1}-1 \qquad (6)$$

rearranging: $(k+1)S_k[n]=\left(n+1\right)^{k+1}-1-\sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!}S_j[n]\right)$

and finally dividing by (k+1) and using Binomial Theorem on $\left(n+1\right)^{k+1}$ we get a recursive formula:

$$\boxed{\sum_{i=1}^{n} i^k = \frac{\sum_{l=0}^{k+1}\left(\frac{(k+1)!}{(k+1-l)!\,l!}n^l\right)-1-\sum_{j=0}^{k-1}\left(\frac{(k+1)!}{(k+1-j)!\,j!}S_j[n]\right)}{k+1}}$$

Where $S_j[n]$ terms in the sum are the previously found formulas for $\sum_{i=1}^{n} i^j$.

For example, the derivation for 4th power (given the formulas for lower powers):

$$\sum_{i=1}^{n} i^4 = \frac{\sum_{l=0}^{5}\left(\frac{5!}{(5-l)!\,l!}n^l\right)-1-\sum_{j=0}^{3}\left(\frac{5!}{(5-j)!\,j!}S_j[n]\right)}{5} =$$

$$\frac{\cancel{1}+5n+10n^2+10n^3+5n^4+n^5-\cancel{1}-\left(S_0[n]+5S_1[n]+10S_2[n]+10S_3[n]\right)}{5} =$$

$$\frac{5n+10n^2+10n^3+5n^4+n^5-\left(n+5\frac{n(n+1)}{2}+10\frac{n(n+1)(2n+1)}{6}+10\frac{n^2(n+1)^2}{4}\right)}{5} =$$

$$\frac{30n+60n^2+60n^3+30n^4+6n^5-6n-15n(n+1)-10n(n+1)(2n+1)-15n^2(n+1)^2}{30} =$$

$$\frac{30n+60n^2+60n^3+30n^4+6n^5-6n-15n-15n^2-20n^3-30n^2-10n-15n^4-30n^3-15n^2}{30} =$$

$$\frac{6n^5+(30-15)n^4+(60-20-30)n^3+(60-15-30-15)n^2+(30-6-15-10)n}{30} =$$

$$\frac{6n^5+15n^4+10n^3-n}{30} = \frac{n(2n+1)(n+1)(3n^2+3n-1)}{30}$$

Binomial Theorem

Binomial Theorem gives a general polynomial expansion for $(a+b)^n$:

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} a^{n-k} b^k$$

(n! is called factorial of n, for definition see the bottom of the page)
The theorem is usually proved using induction and the proof is not included here.

Examples of use of Binomial Theorem:

$$(a+b)^2 = \frac{2!}{(2-0)!0!} a^{2-0} b^0 + \frac{2!}{(2-1)!1!} a^{2-1} b^1 + \frac{2!}{(2-2)!2!} a^{2-2} b^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = \frac{3!}{(3-0)!0!} a^{3-0} b^0 + \frac{3!}{(3-1)!1!} a^{3-1} b^1 + \frac{3!}{(3-2)!2!} a^{3-2} b^2 + \frac{3!}{(3-3)!3!} a^{3-3} b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Factorials

n! is called a factorial of n and is defined as follows:
$n! =$ n×(n-1)×(n-2)×...×(1) for n>0

Example: $3! = 3 \cdot 2 \cdot 1 = 6$

Some properties of factorials:
$n! = n \cdot [(n-1)!]$
1!=1
0!=1
(-x)!=0, where x>0